University of Trento

Department of Physics

Bachelor's Degree in Physics

De Sitter Space

Supervisor
Prof. Luciano Vanzo

Graduant
Marianna Foschi

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Abstract

In this work I will present and describe de Sitter space. I will start with a brief historic introduction, to motivate the importance that this solution of Einstein's equations had in the discussion about cosmological models and the issue concerning the cosmological constant and inflation. Following, I will give a definition of the space and discuss different sets of coordinates that can be used to describe it. Then I will move on to draw Penrose diagrams to describe its causal structure and, finally, I will show by direct calculations that this space is actually a solution to Einstein's field equations. In this discussion I will always refer to the four-dimensional dS space of signature $(1, 3)$, for its relation to the four-dimensional spacetime that we know; however, many results can be extended to higher dimensions.

Contents

1 From Einstein's equations to inflation: de Sitter popping up here and there

In 1916 Einstein published the general theory of relativity[1], which describes gravitation as a result of the geometric structure of spacetime: the motion of free objects is determined by the metric of spacetime which, in turn, is determined by the distribution of matter and energy through the universe. This last assumption is put into mathematical terms by Einstein's field equations:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

that relate the Einstein tensor, which is a function of the metric and its derivatives, to the stress-energy tensor, which represents the density of energy and matter in space. In particular, all the information about the curvature of spacetime is contained in the Riemann tensor, defined as:

$$R^{\rho}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} \tag{2}$$

where the Christoffel symbols are built from the metric in the following way:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2}g^{\mu\gamma}\left(\partial_\rho g_{\sigma\gamma} + \partial_\sigma g_{\gamma\rho} - \partial_\gamma g_{\rho\sigma}\right) \tag{3}$$

The Ricci tensor is obtained by contracting the Riemann tensor on the first and third index:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \tag{4}$$

while if we contract it again we obtain the Ricci scalar:

$$R = R^{\mu}_{\mu} \tag{5}$$

Any possible spacetime that general relativity allows to exist must have a metric $g_{\mu\nu}$ such that eq. (1) is satisfied for some $T_{\mu\nu}$[1].

1.1 From a static to an expanding universe

One year later, Einstein tried to find a static solution to the equations[2], to describe the large scale structure of the universe, which he believed to be static as well. The simplest way to do this was to add a free constant parameter Λ in eq. (1), which we call cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6}$$

This equation allows a solution with $T_{\mu\nu} \neq 0$, called Einstein static universe, which is a spacetime with the topology of $\boldsymbol{R} \times S^3$ and has metric:

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \tag{7}$$

[1] In the following discussion, for the sake of simplicity and readability, I will always use units in which $c = 1$ and omit all the factors c.

We can model the content of matter and energy of the universe as a perfect fluid, so that:

$$T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p\, g_{\mu\nu} \tag{8}$$

where ρ is the energy density, p the pressure and U_μ the 4-velocity of the fluid[2]. Then for $\Lambda > 0$, the field equations can be solved for $p \leq 0$. Einstein static universe is a solution for:

$$p = 0 \tag{9}$$

$$\rho = \frac{\Lambda}{8\pi G} \tag{10}$$

Shortly after, however, Willem de Sitter found a vacuum solution to the field equations with a coordinate system, *static coordinates*, in which the metric[3]:

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\omega_2^2 \tag{11}$$

is time-independent, thus apparently static[3]. Here L is a free constant with the dimension of a length. In the following sections I will explain its meaning in the context of the de Sitter space, as the radius of the spatial universe and its relationship with the cosmological constant Λ. This metric solved the field equations for $T_{\mu\nu} = 0$, challenging the relationship between mass-energy distribution and the geometry of spacetime. However, the curves of constant spatial coordinates are not geodesics, which means that test particles initially at rest do not remain static in such a coordinate system: this universe is actually expanding. It is easier to see this if we rewrite the solution in different coordinates, *global cosmological coordinates*, such that the metric becomes:

$$ds^2 = -dt^2 + L^2 \cosh^2\left(\frac{t}{L}\right) d\omega_3^2 \tag{12}$$

Now the curves of constant spatial coordinates are time-like geodesics (comoving coordinates), but the spatial terms of the metric are changing with time, shrinking and expanding.

In the meanwhile, astronomers were conducting measures on the radiation coming from extra-galactic objects, most of which appeared to be redshifted. This was strong evidence for the expansion of the universe.

1.2 Friedmann-Robertson-Walker universes

An expanding universe, would have been compatible with a de Sitter solution and would not require a cosmological constant anymore. In fact, Einstein's

[2] A perfect fluid is homogeneous and isotropic, so that it can be completely characterized by its rest frame density ρ and pressure p. These symmetry hypothesis require that the stress-energy tensor associated with a perfect fluid is diagonal in the rest frame.

[3] In the following discussion, sometimes, I will use abbreviations $d\omega_n^2$ for the metric over n-spheres.

equations allows for a class of solutions with time-dependent metric coefficients, Robertson-Walker metrics:

$$ds^2 = -dt^2 + R^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2 d\omega_2^2\right] \tag{13}$$

where we can see that the extension of the universe scales according to the cosmic scale factor $R(r)$. RW metrics describe spacetimes that admit a foliation into 3-dimensional maximally symmetric spacelike hyperspaces Σ : the spacetime manifold can be written as $M = \boldsymbol{R} \times \Sigma$. Since Σ is maximally symmetric, its curvature, and thus its curvature scalar $R^{(3)}$, will be constant everywhere. This means that, except from a scale factor, Σ is uniquely determined by the normalized curvature scalar $k \propto R^{(3)}$, that can be equal to 1, 0, -1 if the curvature is respectively positive, null or negative. Spatially maximally symmetric spacetimes are interesting to study as cosmological models because we expect the universe to be isotropic (from experimental evidence) and homogeneous (by assuming the Copernican principle) on the large scale. By inserting eq. (13) and eq. (8) in eq. (1), we obtain that a RW metric is a solution of the field equations if the following conditions (Friedmann equations) for the time evolution of the scale factor hold:

$$H^2(t) := \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \tag{14}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \tag{15}$$

To proceed forward, we have to specify the relationship between ρ and p, that is the equation of state of the perfect fluid. It is realistic to set the density proportional to the pressure:

$$p = w\rho \tag{16}$$

with $|w| \leq 1$[4]. The equation of state, together with the conservation of energy equation:

$$\nabla_\mu T^\mu_\nu = 0 \tag{17}$$

leads to the relation:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R} \tag{18}$$

that can be integrated to obtain:

$$\rho \propto R^{-3(1+w)} \tag{19}$$

The energy-matter content of the universe is very diverse: the energy density ρ is actually a sum of the energy density from different sources. However, we can model it with just three different perfect fluids[4]:

[4]If the source of energy is a perfect fluid, this condition is equivalent to stating that $\rho \geq |p|$ or $p = -\rho \wedge \rho < 0$. This is known as Null Dominant Energy Condition and allows for both a positive and negative vacuum energy density, but only if the latter is balanced by the pressure.

matter collisionless, non relativistic particles with negligible pressure. It is characterized by $p_M = 0$, $w_M = 0$, $\rho_M \propto R(t)^{-3}$.

radiation electromagnetic radiation or ultrarelativistic particles, characterized by $p_R = \frac{1}{3}\rho_R$, $w_R = 1/3$, $\rho_R \propto R(t)^{-4}$

vacuum energy also known as cosmological constant. If the vacuum has a non 0 energy density, it seems reasonable to assume that it is homogeneous, so the associated stress-energy tensor will be proportional to the metric, this makes it indistinguishable from the cosmological term in the field equations[5]. It follows that if we represent it as a perfect fluid we get $p_\Lambda = -\rho_\Lambda$, $w_\Lambda = -1$, $\rho_\Lambda \propto R(0) = \text{const}$.

curvature this is not actually an energy density, but we can treat the curvature term in Friedmann's equation as if it was, defining $\rho_k = -\frac{3k}{8\pi G R(t)^2}$. It is then characterized by $w = -1/3$, $\rho \propto R(t)^{-2}$

If we assume that one kind of energy density dominates over the others at a certain time, and that the spatial curvature is null, we can combine the relations between densities and scale factor with Friedmann's equations, to get:

$$\dot{R}(t) \propto R(t)^{-\frac{1}{2}(3w+1)} \tag{20}$$

that integrated in time becomes:

$$R(t) \propto \begin{cases} t^{\frac{2}{3}(w+1)} & \text{if } w \neq -1 \\ e^t & \text{if } w = -1 \end{cases} \tag{21}$$

For a vacuum dominated universe, the scale factor expands like an exponential and the metric becomes:

$$ds^2 = -dt^2 + e^{Ht}\left[dx^2 + dy^2 + dz^2\right] \tag{22}$$

which is again the de Sitter space metric written in an other set of coordinates, *inflationary coordinates*. It is interesting that de Sitter can be a model for a flat ($k = 0$), empty ($\rho_M = \rho_R = 0$) universe whose expansion is driven by a positive cosmological constant.

1.3 Inflation

Is a vacuum dominated universe a realistic model for our universe? From eq. (21) we see that the scale factor grows in time for all the different kind of matter-energy that we examined before. Interpolating this trend in the past, we can assume that there was a time at which $R = 0$. Also, the energy density of different sources scales differently with the scale factor. If we assume that initially both matter, radiation and vacuum energy were present, as the scale

[5] In this text, the terms *cosmological constant* and *vacuum energy density* are interchangeable.

factor expands, the radiation term will be the one to die out first in Friedmann's equation, followed by the matter term, while the vacuum term will remain constant. We speak then about a radiation dominated era, a matter dominated era and finally a vacuum dominated era. This is the so-called Big Bang theory, which is successful in explaining many observations. Current experimental data tells us that today we are around the end of a matter dominated era, so that de Sitter space could be a large scale model for the future universe.

In addition to this, vacuum energy takes on an important role in other era of the universe's history. The simple Big Bang model does not provide an explanation to the highly non generic conditions in which we find the current universe to be: almost 0 spatial curvature and isotropy of the cosmic microwave background radiation (CMB). Let's start by examining the flatness problem. If we suppose that the vacuum energy is 0, the first Friedmann equation (eq. (14)) becomes:

$$H(t)^2 = \frac{8\pi G}{3}\left(\rho_M + \rho_R\right) - \frac{k}{R(t)^2} \tag{23}$$

$$= \frac{8\pi G}{3}\left(\rho_{M0} R(t)^{-3} + \rho_{R0} R(t)^{-4}\right) - kR(t)^{-2} \tag{24}$$

The curvature term decreases more slowly than the other two, so we would expect that, at present time, the ratio between the curvature term and energy term would be much greater than one, which is not. The second issue has to do with a horizon problem. In the context of a universe that has been existing for a limited amount of time, there is a maximal distance that photons may have travelled since the Big Bang, we call it a particle horizon. In a matter or radiation dominated universe, the physical distance between two points grows as $R(t)$, while the physical horizon grows faster ($\sim R(t)^{n/2}$, $n = 3, 4$). Observations show that the CMB is isotropic in areas of the sky that are outside each others' particle horizon, which means that they were not causally connected at the time the radiation formed.

The two problems can be solved by postulating a period of inflation, that is a period of accelerating expansion ($\ddot{R}(t) > 0$). From the second Friedmann equation, we see that this is possible in the case of a non 0 vacuum energy, that drives an exponential expansion of the space. The flatness problem is solved because, during the inflation period, the density term remains constant, while the curvature term rapidly decreases, providing an explanation to why they are of the same order today. At the same time, also the horizon problem is solved, because the physical horizon size would be much bigger that the one estimated without inflation. So an expanding, vacuum dominated universe may have characterized a past phase of our spacetime.

2 Coordinate systems: the various appearance of de Sitter space

Let's now see how all the coordinates systems that we mentioned before are actually describing the same manifold.

2.1 Cartesian coordinates

We can start by defining de Sitter space as an hyperboloid H with radius L, of equation:

$$-\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2+\left(X^4\right)^2=L^2 \tag{25}$$

embedded in a five-dimensional Minkowsky space with metric:

$$ds^2=-\left(dX^0\right)^2+\left(dX^1\right)^2+\left(dX^2\right)^2+\left(dX^3\right)^2+\left(dX^4\right)^2 \tag{26}$$

Eq. (25) tells us that H is the locus of points that have the same invariant distance from the origin. So any transformation from the ten-parameter Lorentz group in five dimension will leave H unchanged: they are automorphisms of H. Such a transformation will also leave the metric on H, which is induced from the 5-dimensional Minkowsky one, unchanged; therefore H is a maximally symmetric space.

The hyperboloid H, embedded in Minkowsky space, is related to the hypersphere:

$$\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2+\left(X^4\right)^2=L^2 \tag{27}$$

embedded in Euclidean space, by the linear and homogeneous transformation $X^0 \to -iX^0$. The geodesics on the hypersphere are great circles, defined by the intersection of the hypersphere with hyperplanes passing through the origin. The corresponding geodesics on H are, in the same way, the plane curves defined as the intersections between the hyperboloid and hyperplanes through the origin[5]. If we assign to X^0 the meaning of time coordinate, null geodesics are defined by hyperplanes that form an angle θ with the X^0 axis equal to $\pi/4$, spacelike geodesics have $\theta > \pi/4$, timelike geodesics have $\theta < \pi/4$. In fig. 1 a reduced model is shown, were the coordinates X_3 and X_4 have been removed[6]. In particular, we can interpret the reduced model as showing only the time coordinate and one space coordinate, so that every point of the reduced H represents a 2-sphere (S^2) at a certain time. This model allows us to visualize spacelike geodesics as ellipses with the semi-minor axis laying in the $X^0 = 0$ plane, timelike geodesics as hyperbolas and null geodesics as straight lines. All of this curves are either closed curves or extend to infinity, meaning that H is geodesically complete[7].

[6] I will continue to use this model to draw pictures of the different coordinate systems and to explain concepts that will later be extended to the full model.

[7] This property is valid also in the extended model.

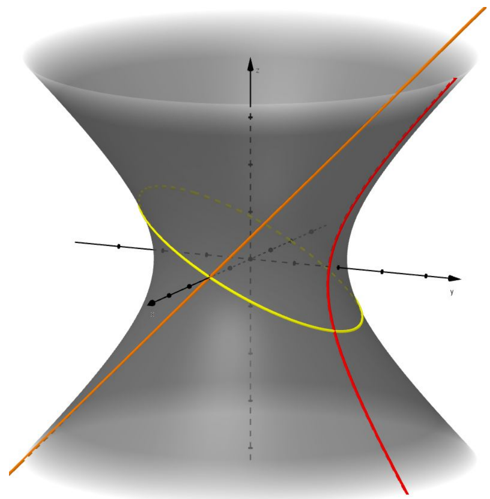

Figure 1: Reduced model for de Sitter space. The yellow ellipse is a spacelike geodesic, the orange line is a null geodesic, while the red hyperbole is a timelike one.

2.2 Global cosmological coordinates

Even though the X^i coordinates are useful to visualize geodesics and Lorentz transformations, H is a four-dimensional manifold and it can be described by only 4 coordinates. We can eliminate the redundant coordinate by introducing global cosmological coordinates (t, χ, θ, ϕ) such that:

$$\begin{cases} X^0 &= L\sinh\left(\frac{t}{L}\right) \\ X^1 &= L\cosh\left(\frac{t}{L}\right)\cos\chi \\ X^2 &= L\cosh\left(\frac{t}{L}\right)\sin\chi\cos\theta \\ X^3 &= L\cosh\left(\frac{t}{L}\right)\sin\chi\sin\theta\cos\phi \\ X^4 &= L\cosh\left(\frac{t}{L}\right)\sin\chi\sin\theta\sin\phi \end{cases} \tag{28}$$

These expression satisfy eq. (25) and, by applying the change of coordinates to the metric in eq. (26), we obtain the metric:

$$ds^2 = -dt^2 + L^2\cosh^2\frac{t}{L}\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{29}$$

where the term in round brackets is the metric over a two-sphere $d\omega_2^2$ and the term in square brackets is the metric over a three-sphere $d\omega_3^2$. In this way we introduce a coordinate singularity that has the same nature of the singularity of spherical coordinates, it is not a problem we should care about because it does not reflect a real singularity of the manifold and global cosmological coordinates cover the whole manifold.

In this parametrization of the hyperboloid, time runs from $-\infty$ to $+\infty$ and sections of constant time coordinate are 3-spheres with a time dependent radius that initially shrinks from an infinite value and then re-expands to infinity. If we consider only the half of the manifold with $t \geq 0$, the model describes an expanding universe. For:

$$t \in (-\infty, +\infty) \quad , \quad \theta \in [0, 2\pi] \quad , \quad \chi, \phi \in [0, \pi] \tag{30}$$

the coordinates cover the whole manifold, therefore the topology of H is $\boldsymbol{R} \times S^3$, where S^3 is the three-sphere, meaning that the de Sitter space is conformal to a part of Einstein's static universe[8]. Fig. 2 shows curves of constant coordinates

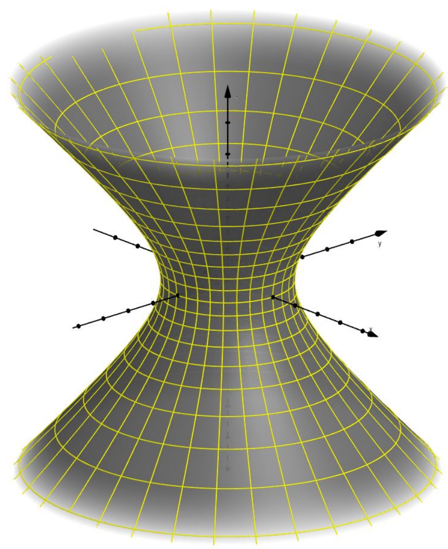

Figure 2: Curves of constant coordinates in the global cosmological system, reduced model.

for the reduced model. From the picture is it easy to see that curves of constant spatial coordinates are timelike geodesics, which means that if we put test particles at rest in these coordinates, they will remain at rest in the coordinate system and follow the expansion of the space. A coordinate system with this property is called *comoving*.

This representation, however, highlights an unpleasant feature of the global cosmological coordinates. The sections of constant time, that should represent the space at time t, are 1-spheres (in the reduced model), but only the one with $t = 0$ is a geodesics, while the others are not. Lorentz transformations of the space of simultaneity $t = 0$ cannot transform it in one of the other spaces of simultaneity and vice versa. This means that there is an essential inequivalence between spaces at constant time.

[8]A conformal transformation is one on the form $ds'^2 = \omega^2(x) ds^2$ with $\omega \neq 0 \; \forall$ points x. It consist in nothing more than a local change of scale.

2.3 Static coordinates

So we would like to develop a coordinate system in which spaces of constant time are built from families of geodesic curves. In section 2.1 we explained what spacelike geodesics look like in the reduced model. We can consider the family

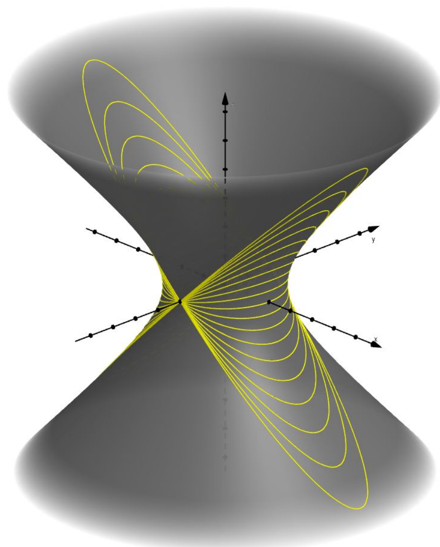

Figure 3: Family of ellipses that are sections of constant time in the reduced model.

of ellipses ϵ_t shown in fig. 3 and identify each space of constant t with a function of the inclination of the plane that defines the geodesic section. For example it is useful to choose:

$$\tanh\left(\frac{T}{L}\right) = \frac{X^0}{X^1} \tag{31}$$

to define the time coordinate T. Then eq. (25) is satisfied by choosing so-called static coordinates (T, χ, θ, ϕ) such that:

$$\begin{cases} X^0 &= L\cos\chi\sinh\left(\frac{T}{L}\right) \\ X^1 &= L\cos\chi\cosh\left(\frac{T}{L}\right) \\ X^2 &= L\sin\chi\cos\theta \\ X^3 &= L\sin\chi\sin\theta\cos\phi \\ X^4 &= L\sin\chi\sin\theta\sin\phi \end{cases} \tag{32}$$

and the metric becomes:

$$ds^2 = -\cos^2\chi dT^2 + L^2 d\chi^2 + L^2\sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{33}$$

(it is the same as eq. (11) if we identify: $r/L \to \sin\chi$, $t \to T$). The characteristic feature of the static coordinates is that they show time independent metric coefficients. This seems to suggest that de Sitter space can be a model for a static universe. We should note however that curves of constant spatial coordinates now are not geodesics (except the one for $\chi = 0$, $\theta = 0$, $\phi = 0$), so test particles put initially at rest, will soon appear to be moving in the coordinate reference frame. Furthermore, with such a choice of coordinates, a singularity arises (due only to coordinates, not to a real feature of the manifold): on the two-spheres determined by $X^0 = 0$ and $X^1 = 0$, the time becomes indeterminate, because all spaces of simultaneity meet (we see this also from the first of eq. (32) when $X^0 = 0$ and $\chi = \pm\pi/2$). This two-dimensional surface is called *bifurcate horizon* and it separates the parts of H in which $|\chi| > \pi/2$, from those where $|\chi| < \pi/2$.

A second problem is that, even if we let T range over all $\boldsymbol{R}$ and χ run from $-\pi$ to $+\pi$, the static coordinates will not cover the whole manifold, but only the two saddle shaped regions in fig. 4 (or just one if we set the limitation $|\chi| \leq \pi/2$) that are determined by the condition:

$$\left(X^2\right)^2 + \left(X^3\right)^2 + \left(X^4\right)^2 \leq L^2 \tag{34}$$

together with eq. (25). The boundary of this region consist of the null three-

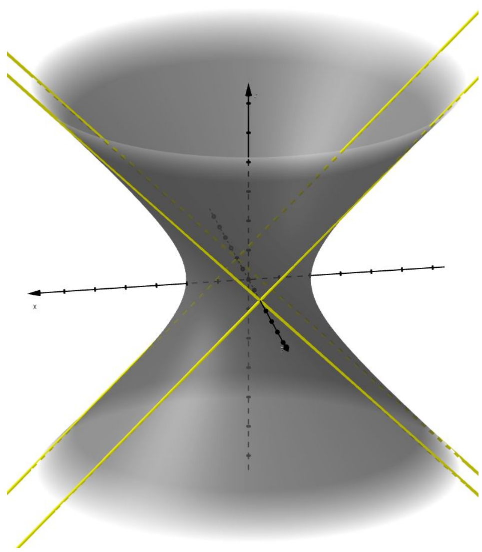

Figure 4: Saddle shaped regions covered by static coordinates.

dimensional surface drawn by light rays spreading in the future or in the past from the points of the bifurcate horizon.

Since timelike geodesics have an ascent steeper than $\pi/4$, there are many of them that have only a finite section inside the saddle region, while other reach past or future infinity inside the region, but exit or enter it at a finite eigen-time. Referring to the reduced model, only one timelike geodesic is completely contained in the region: the one for $R = 0$, which is the only observer in these coordinates that is both static and sitting on a geodesic. From the point of view of the static frame, particles that exit the saddle regions are seen as approaching the mass horizon indefinitely, reaching it only for $t \to \infty$. At first this sound like a paradox, because the mass horizon is connected to every point inside the region by a spacelike geodesic, however, as the particle is approaching the null surface of the boundary, its invariant distance from the mass horizon, though being spacelike, tends to 0.

Curves of constant coordinates are shown in fig. 5.

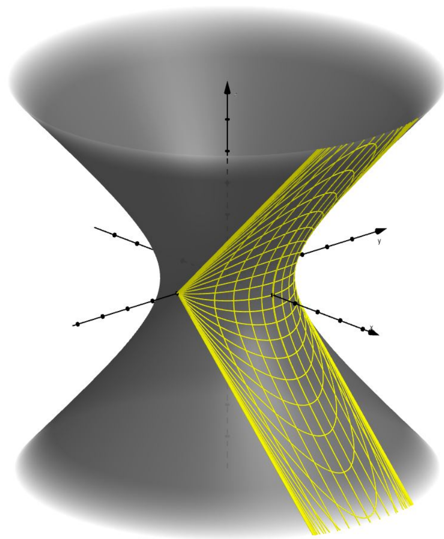

Figure 5: Curves of constant coordinates in the static system, reduced model.

2.4 Inflationary coordinates

Both in the global cosmological and in the static coordinates the spaces of simultaneity, which are the sections of H with constant time, are bounded regions for each value of the time coordinate. Now, we turn to a coordinate system, named inflationary coordinates, in which the spaces of simultaneity are unbounded. For simplicity, I will first develop the coordinates in the reduced model and then extend them to the full one.

We define as spaces of simultaneity the spacelike sections cut in H by a sheaf of parallel planes with an inclination of $\pi/4$ with respect to X^0. This correspond to a family of parabolas, for example those parallel to the X^2 axis, each of which is identified by a different constant value:

$$X^0 + X^1 = \text{const.} \tag{35}$$

The time parameter will then be of the form:

$$t = f\left(X^0 + X^1\right) \tag{36}$$

We look for a spatial coordinate that is perpendicular to the time one. On H, an infinitesimal displacement with the condition $t =$ const. is given by differentiating eq. (25) and eq. (35):

$$\begin{cases} dX^0 + dX^1 = 0 \\ -X^0 dX^0 + X^1 dX^1 + X^2 dX^2 = 0 \end{cases} \tag{37}$$

from which follows:

$$dX^2 : dX^1 : dX^0 = -(X^0 + X^1) : X^2 : -X^2 \tag{38}$$

A direction $(\delta X^0, \delta X^1, \delta X^2)$ Minkowsky orthogonal to (dX^0, dX^1, dX^2) must satisfy the condition:

$$-\left(X^0 + X^1\right)\delta X^2 + X^2 \delta X^1 + X^2 \delta X^0 = 0 \tag{39}$$

$$\frac{\delta(X^0 + X^2)}{X^0 + X^2} = \frac{\delta X^2}{X^2} \tag{40}$$

$$\frac{X^2}{X^0 + X^1} = \text{const.} \tag{41}$$

So we choose the spatial coordinate:

$$r = g\left(\frac{X^2}{X^0 + X^1}\right) \tag{42}$$

Now we extend the result to the full model and choose the functions f and g:

$$\begin{cases} t = L \log\left(\frac{X^0 + X^1}{L}\right) \\ x = L\frac{X^2}{X^0 + X^1} \\ y = L\frac{X^3}{X^0 + X^1} \\ z = L\frac{X^4}{X^0 + X^1} \end{cases} \tag{43}$$

We used a logarithmic function for the time coordinate, so to push away to $t \to \infty$ the degenerate space of simultaneity at $X^0 + X^1 = 0$ that consist of two parallel hyperplanes, instead of a spacelike paraboloid. In this way, though, we cannot cover the whole manifold, but only the half of the points of H for which $X^0 + X^1 > 0$. In fig. 6 we can see curves of constant coordinate for the reduced model.

This time, the spatial coordinates x, y, z are not bounded, but range from $-\infty$ to $+\infty$, in particular, for $t \to -\infty$, also $x, y, z \to \infty$. The inverse coordinate transformation is:

$$\begin{cases} X^1 + X^0 & = Le^{\frac{t}{L}} \\ X^1 - X^0 & = Le^{-\frac{t}{L}} - \frac{x^2+y^2+z^2}{L} e^{\frac{t}{L}} \\ X^2 & = xe^{\frac{t}{L}} \\ X^3 & = ye^{\frac{t}{L}} \\ X^4 & = ze^{\frac{t}{L}} \end{cases} \tag{44}$$

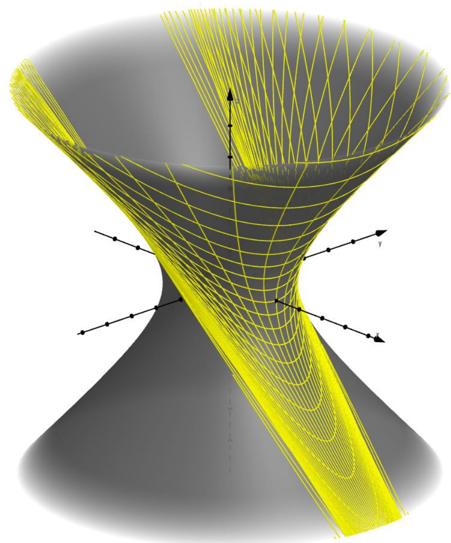

Figure 6: Curves of constant coordinates in the inflationary system, reduced model.

from which we can calculate the metric:

$$ds^2 = -dt^2 + e^{2\frac{t}{L}} \left[dx^2 + dy^2 + dz^2\right] \tag{45}$$

The expression inside the square brackets is the metric over a flat Euclidean space, with the topology of $\boldsymbol{R}^3$, while the coefficient $e^{2\frac{t}{L}}$ grows exponentially with time, meaning that the space, which is already unbounded, is also expanding. An interesting feature of this system is that the planes $\frac{X^i}{X^0+X^1} = \text{const.}, i = 2, 3, 4$ that cut on H the curves of constant spatial coordinates, contain the origin, so these curves are geodesics. This means that test particles initially at rest, will remain at rest respect to the spatial coordinates, even though their respective distances will grow exponentially. Particles that move along a geodesic, but are not at rest in this coordinate system, will asymptotically approach a meridian hyperbola (curves of constant spatial coordinates in the global cosmological frame) which means that its coordinate velocity will tend asymptotically to 0, no matter what the initial value was, even if it was c.

2.5 Conformal coordinates

The last coordinate system to which we turn is just a reparametrization of the global cosmological coordinates. Actually, the spatial conformal coordinates are exactly the same as the the global ones. The relationship between the conformal time coordinate T and the cosmological time t is:

$$\cosh\left(\frac{t}{L}\right) = \frac{1}{\cos T} \tag{46}$$

with:

$$T \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \tag{47}$$

so that the conformal coordinates are:

$$\begin{cases} X^0 & = L \tan T \\ X^1 & = L \frac{1}{\cos T} \cos\chi \\ X^2 & = L \frac{1}{\cos T} \sin\chi \cos\theta \\ X^3 & = L \frac{1}{\cos T} \sin\chi \sin\theta \cos\phi \\ X^4 & = L \frac{1}{\cos T} \sin\chi \sin\theta \sin\phi \end{cases} \tag{48}$$

The usefulness of the transformation is that the domain of time becomes bounded. We can use this coordinate system, for example, to draw Penrose diagrams of de Sitter space, in which we can visualize the whole space (actually, inevitably just a two or three-dimensional section) in a finite portion of the plane.

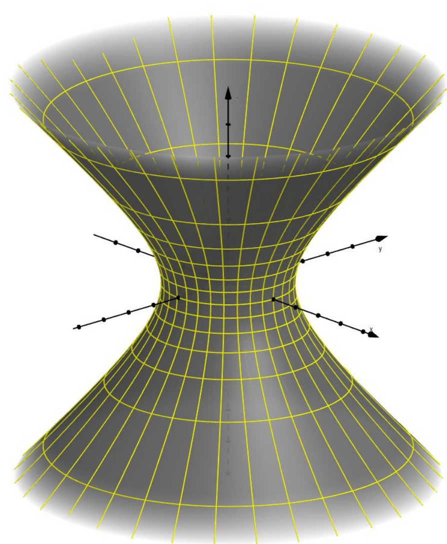

Figure 7: Curves of constant coordinates in the conformal system, reduced model.

Differentiating both sides of eq. (46) we get:

$$\frac{1}{L}\sinh\frac{t}{L}dt = \frac{\sin T}{\cos^2 T}dT \Rightarrow dt = L\frac{\sin T}{\cos^2 T}\sinh^{-1}\left(\frac{t}{L}\right)dT \tag{49}$$

Now we can calculate the metric starting from eq. (29):

$$ds^2 = \frac{L^2}{\cos^2 T}\left[-dT^2 + \left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\right] \tag{50}$$

that is the same as Einstein's static universe metric up to the multiplicative factor $L^2/\cos^2 T$. In fig. 7 we see curves of constant coordinates.

3 Causal structure: de Sitter on a small square

3.1 Penrose diagrams

To study infinity of spacetimes one uses, after Penrose, conformal techniques, which also provide a simple way to represent curved spacetimes, at least in two dimensions. This means that we need to find an appropriate coordinate system that satisfies the following requirements:

- it must include a timelike coordinate and a spatial one;
- light rays in the direction of the space coordinate shown are represented by 45° lines in every point of the diagram;
- timelike, spacelike and null infinity are located at finite values of the coordinates, so the whole manifold can be represented in a limited diagram.

Spacetime diagrams of this kind are called *conformal diagrams* or *Penrose diagrams*.

In order to build one for de Sitter space, we can start by considering the representation of the space in conformal coordinates:

$$ds^2 = \frac{L^2}{\cos^2 T}\left[-dT^2 + \left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\right] \tag{51}$$

where

$$T \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \quad , \quad \chi \in [0, \pi] \tag{52}$$

Light rays in the direction ∂_χ are characterized by $ds^2 = 0$ and $d\theta = d\phi = 0$, so their coordinate velocity is:

$$\frac{d\chi}{dT} = 1 \tag{53}$$

Conformal coordinates satisfy all the requirements. The topology of the space now is $\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times S^3$, so a possible way to visualize it is to represent it as a section of height π of a cylinder of unitary radius (see fig. 8),where each point of the surface represents a two-sphere. If we unroll this area, we obtain the conformal spacetime diagram for de Sitter space (fig. 9) where $\chi = 0$ and $\chi = \pi$ are identified. In this representation we keep only time and radial coordinates (T, χ), with a reduced metric:

$$ds^2 = \frac{L^2}{\cos^2 T}\left[-dT^2 + d\chi^2\right] \tag{54}$$

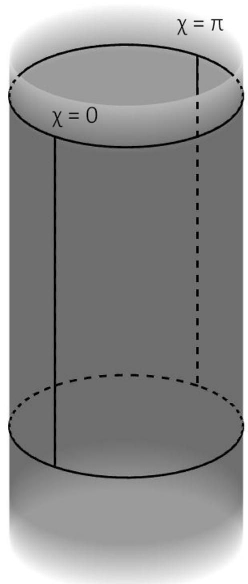

Figure 8: Representation of de Sitter space as a patch on Einstein's static universe space.

In fig. 10, instead, the conformal diagram of the portion of de Sitter that is covered by static coordinates is shown. It appears evident here that the singularity on the border is only due to the choice of coordinates.

3.2 Causal structure

One of the main reason we draw Penrose diagrams is to study infinity, which is represented by the conformal boundary. In our case, the diagram shows that de Sitter space has both past and future timelike infinities ($\mathcal{I}^+$ and $\mathcal{I}^-$), on which all non spacelike curves respectively start and end. This characteristic is different for example from Minkowsky space, where the conformal boundary is lightlike (see fig. 11).

I will give some definitions that will result useful in the discussion about the causal structure[6]. For a given point p in the conformal diagram, we indicate with:

chronological future (past) $I^{\pm}(p)$ the set of points that can be reached from p with a future (past) directed timelike curve. In simple terms, it is the interior of the null cone spreading from p.

causal future (past) $J^{\pm}(p)$ the set of points that can be reached from p with

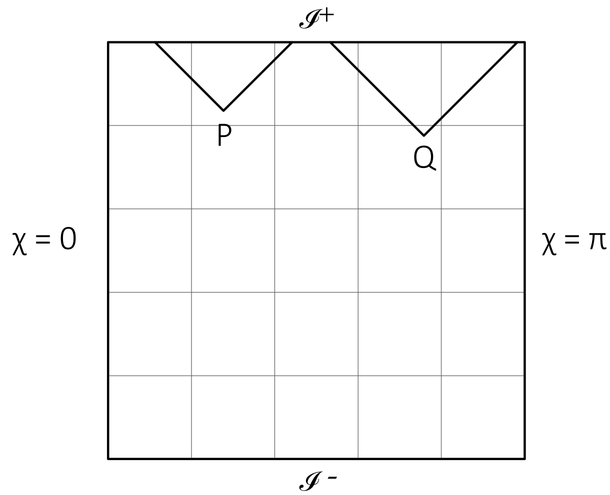

Figure 9: Penrose diagram of de Sitter space. Curves of constant conformal coordinates are shown, along with the future-directed light cones of two points.

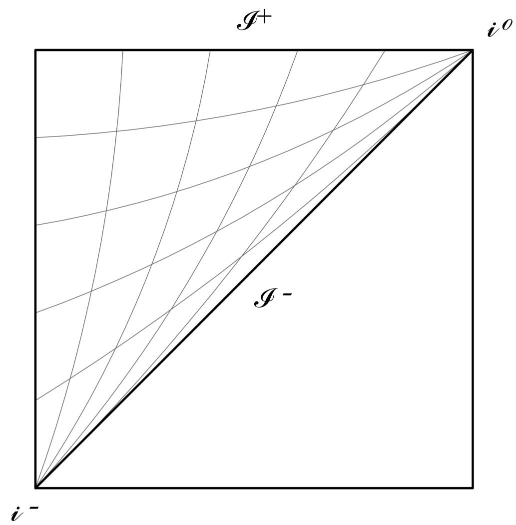

Figure 10: Penrose diagram of de Sitter space in the static frame. Curves of constant static coordinates are shown.

> a future (past) directed non spacelike curve. It is the interior and the boundary of the null cone spreading from p.

Now we consider a family of observers that move along timelike geodesics (for example they can be the family of curves with constant spatial coordinates in

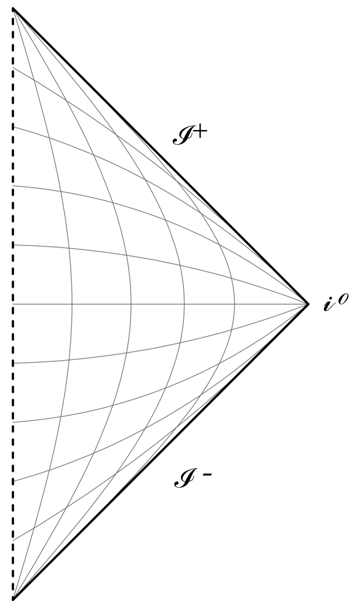

Figure 11: Penrose diagram of Minkowsky space. Curves of constant coordinates are shown.

conformal or inflationary coordinates). If p is a point on the worldline of observer O, corresponding to a value t of its eigentime, we say that another observer $\bar{O}$ has been seen by O if $\bar{O}$'s worldline crosses O's causal past at p (fig. 12). We define:

particle horizon for O at p, $H(O, p)$, the timelike surface that divides the worldlines of observers seen by O at p, and those that have not been seen (yet).

On the conformal diagram all the worldlines are limited in the past and in the future by points, say q_p and q_f, on $\mathcal{I}^-$ and $\mathcal{I}^+$. We define:

future event horizon of the worldline, $E^+(O)$, as the null surface that divides the points of events ever seen by O, from those that O never sees. We can think of it as $J^-(q_f) - I^-(q_f)$.

past event horizon of the worldline, $E^-(O)$, as the null surface that divides the points of events from which is possible to see O at some time, from those that never see O. We can think of it as $J^+(q_p) - I^+(q_p)$.

Event horizons and particles horizons are not present in all spacetimes (for example they do not exist in Minkowsky space), but in de Sitter there are both. Every worldline has both a past and a future event horizon, while the area inside the particle horizon will eventually grow to cover the whole space. In fact, we can choose two points (like in fig. 9) that have either causal futures

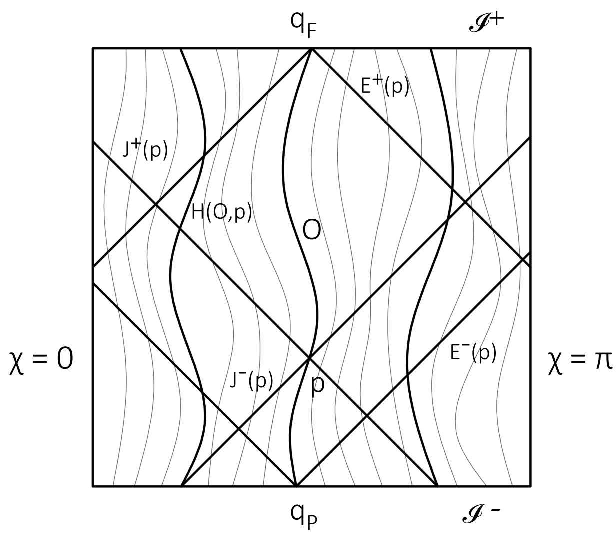

Figure 12: In this Penrose diagram of de Sitter space the worldlines of a family of observers are shown. The light cones that represent the boundary of $J^{\pm}(p)$ are shown, as well as the particle horizon at p and future and past event horizon of O.

or causal pasts completely disconnected. This possibility is a consequence of the future and past infinities being spacelike. On a more physical level, we can interpret this as due to the expansion of the universe. Spatial distances expand at a rate faster that the speed of light, so that observers that were once arbitrarily close, will eventually undertake causally disconnected paths (unless they end exactly at the same point). On the other hand, at early times, in the history of this evolving universe, the spaces of simultaneity can be divided into spacelike patches, some of which have non overlapping causal pasts. Turning to the conformal diagram of de Sitter in static coordinates, we see that past infinity is null, not spacelike. This means that observers do not have a past event horizon, but only a future one, and there is no particle horizon.

4 A solution to Einstein's equations: some long but necessary calculations

I am going to show now that de Sitter space is actually a solution to Einstein's field equations. Referring to eq. (6), I will prove by substitution that the metric in eq. (45), which in matrix form is given by:

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{\frac{2t}{L}} & 0 & 0 \\ 0 & 0 & e^{\frac{2t}{L}} & 0 \\ 0 & 0 & 0 & e^{\frac{2t}{L}} \end{bmatrix} \tag{55}$$

solves the equation for $T_{\mu\nu} = 0$, $\Lambda > 0$ and determine the value of Λ. The field equations reduces to:

$$R_{\mu\nu} = \left(\frac{R}{2} - \Lambda\right) g_{\mu\nu} \tag{56}$$

that means that I just need to prove that the Ricci tensor is proportional to the metric. I choose to use the metric in inflationary coordinates because it is the most easy to work with and the proof does not loose in generality because Einstein's equations are tensor equations, so they are coordinate independent.

Let's start by calculating Christoffel symbols, using eq. (3). Since our metric is diagonal, the sum over the index γ reduces to only one term and becomes:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\mu} \left(\partial_\rho g_{\sigma\mu} + \partial_\sigma g_{\mu\rho} - \partial_\mu g_{\rho\sigma}\right) \tag{57}$$

where μ is now a specific coordinate (we do not sum over it). The only coordinate dependence in the metric is the one on t and the time derivative is:

$$\partial_t g_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{L} e^{\frac{2t}{L}} & 0 & 0 \\ 0 & 0 & \frac{2}{L} e^{\frac{2t}{L}} & 0 \\ 0 & 0 & 0 & \frac{2}{L} e^{\frac{2t}{L}} \end{bmatrix} \tag{58}$$

So the only non-0 Christoffels are:

$$\Gamma^{t}_{\rho\sigma} = -\frac{1}{2} g^{tt} \partial_t g_{\rho\sigma} \tag{59}$$

$$\Gamma^{\rho}_{t\sigma} = \Gamma^{\rho}_{\sigma t} = \frac{1}{2} g^{\rho\rho} \partial_t g_{\sigma\rho} \tag{60}$$

where again we do not sum over ρ nor t. Using again the fact that the metric is diagonal, we can reduce the non-0 Christoffels to:

$$\Gamma^{t}_{ii} = -\frac{1}{2}(-1)\frac{2}{L} e^{\frac{2t}{L}} = \frac{1}{L} e^{\frac{2t}{L}} \tag{61}$$

$$\Gamma^{i}_{ti} = \Gamma^{i}_{it} = \frac{1}{2} e^{-\frac{2t}{L}} \frac{2}{L} e^{\frac{2t}{L}} = \frac{1}{L} \tag{62}$$

where i stands only for the spatial indexes. The Ricci tensor can be written as:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = R^{t}_{\mu t\nu} + R^{i}_{\mu i\nu} \tag{63}$$

Let's calculate each of the two terms on the right side, the first one is:

$$
\begin{aligned}
R^t_{\mu t\nu} &= \partial_t \Gamma^t_{\nu\mu} - \partial_\nu \Gamma^t_{t\mu} + \Gamma^t_{t\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^t_{\nu\lambda}\Gamma^\lambda_{t\mu} && (64)\\
&= \partial_t \Gamma^t_{\nu\mu} - \Gamma^t_{\nu\lambda}\Gamma^\lambda_{t\mu} && (65)\\
&= \delta_{\mu\nu}\left[\frac{1}{L^2}e^{\frac{2t}{L}}\right] - \delta_{\mu t}\delta_{\nu t}\left[\frac{1}{L^2}e^{\frac{2t}{L}}\right] && (66)
\end{aligned}
$$

while the second one, for the coordinate x is:

$$
\begin{aligned}
R^x_{\mu x\nu} &= \partial_x \Gamma^x_{\nu\mu} - \partial_\nu \Gamma^x_{x\mu} + \Gamma^x_{x\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^x_{\nu\lambda}\Gamma^\lambda_{x\mu} && (67)\\
&= -\partial_\nu \Gamma^x_{x\mu} + \Gamma^x_{xt}\Gamma^t_{\nu\mu} - \Gamma^x_{\nu t}\Gamma^t_{x\mu} - \Gamma^x_{\nu x}\Gamma^x_{x\mu} && (68)\\
&= -\delta_{\mu t}\delta_{\nu t}\left(\partial_t \Gamma^x_{xt} + \Gamma^x_{xt}\Gamma^t_{xx}\right) + \delta_{\mu\nu}\Gamma^x_{xt}\Gamma^t_{xx} && (69)\\
&\quad - \delta_{\mu x}\delta_{\nu x}\Gamma^x_{xt}\Gamma^t_{xx} - \delta_{\mu t}\delta_{\nu t}\Gamma^x_{tx}\Gamma^x_{xt} && (70)\\
&= -\delta_{\mu t}\delta_{\nu t}\left[\frac{1}{L^2}e^{\frac{2t}{L}} + \frac{1}{L^2}\right] - \delta_{\mu x}\delta_{\nu x}\left[\frac{1}{L^2}e^{\frac{2t}{L}}\right] + \delta_{\mu\nu}\left[\frac{1}{L^2}e^{\frac{2t}{L}}\right] && (71)
\end{aligned}
$$

Summing up all the terms we get:

$$
R_{\mu\nu} = \delta_{\mu\nu}\left[\frac{3}{L^2}e^{\frac{2t}{L}}\right] - \delta_{\mu t}\delta_{\nu t}\left[\frac{3}{L^2} + \frac{3}{L^2}e^{\frac{2t}{L}}\right] = \frac{3}{L^2}g_{\mu\nu} \tag{72}
$$

So the metric satisfies Einstein's equations. The Ricci scalar is:

$$
R = \frac{3}{L^2}g^{\mu\nu}g_{\mu\nu} = \frac{12}{L^2} \tag{73}
$$

hence, using eq. (56), we can calculate the cosmological constant:

$$
\Lambda = \frac{6}{L^2} - \frac{3}{L^2} = \frac{3}{L^2} \tag{74}
$$

In this model, the cosmological constant is linked to the radius of the universe.

4.1 Symmetry considerations

This brute force calculation could have been made easier by using the fact that de Sitter space is maximally symmetric. Such a space is defined by the property of having the maximum number of independent symmetries[9] possible, that is:

$$
\frac{1}{2}n(n+1) \tag{75}
$$

where n is the dimension of the space. If this is the case, the geometry of the space will be invariant under translation-like and rotation-like transformations. The metric must be invariant as well, implying that the curvature, which is

[9]By symmetry, here, we mean as isometry, which is a symmetry of the metric tensor. A symmetry ϕ of a tensor T is a diffeomorphism such that T is invariant after being pulled back under ϕ.

a scalar built from the metric and its derivatives, is constant through all the manifold. For a given dimension and signature, up to a re-scaling, there are only three different maximally symmetric spaces, characterized by positive, null, or negative curvature. De Sitter space is the maximally symmetric space of positive curvature for the case $n = 4$, signature (1,3).

It can be shown that, if the number of symmetries is maximal, the Riemann tensor can be written as:

$$R_{\rho\mu\sigma\nu} = \frac{R}{n(n-1)} \left(g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma} \right) \tag{76}$$

It follows that for $n = 4$ the Ricci tensor is:

$$\begin{aligned} R_{\mu\nu} &= \frac{R}{n(n-1)} g^{\sigma\rho} \left(g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma} \right) &(77)\\ &= \frac{R}{12} \left(g^{\sigma}_{\sigma} g_{\mu\nu} - g^{\sigma}_{\nu} g_{\mu\sigma} \right) &(78)\\ &= \frac{R}{12} \left(4 g_{\mu\nu} - g_{\mu\nu} \right) &(79)\\ &= \frac{R}{4} g_{\mu\nu} &(80) \end{aligned}$$

from which its easy to show that Einstein's vacuum field equation is satisfied.

5 Conclusions

We have seen that de Sitter space is a vacuum, maximally symmetrical, non globally static solution to Einstein's field equations. It describes a spacetime with constant positive curvature and topology $\boldsymbol{R} \times S^3$, consisting of three-spheres that contract and expand in time. We can describe it with different set of coordinates, which cover the whole manifold or just parts of it. Depending on the time foliation that we choose, the reduced metric on the spaces of simultaneity changes, so that we can have either a flat, infinite space or a positively curved, closed space.

The major importance of de Sitter spacetime in today's cosmology is that it corresponds to a vacuum dominated universe, which means a universe in which the predominant form of energy is vacuum energy (or dark energy if we prefer that interpretation). This is thought to be the future state in which our universe will evolve, as well as an approximation of a past state in which our universe was during the period of primordial inflation.

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